ISSN 2348-1218 (print)
International Journal of Interdisciplinary Research and Innovations ISSN 2348-1226 (online)
Vol. 6, Issue 4, pp: (622-626), Month: October - December 2018, Available at: www.researchpublish.com

Generalizations in Fractals

Vincent N. S.

(Department of Mathematics, Christ College (Autonomous), Irinjalakuda, India)
Author Email Id: vinnncmi@gmail.com

Abstract: **The studies have been mainly done in the discrete dynamical systems in topological spaces. According to Robert Devaney, the three ingredients of chaos are sensitivity, density and transitivity. We answer a series of questions like whether the iterated function system can be chaotic. Will the contractions are Devaney chaotic. If we can find such a chaotic contraction, will it generate a self similar set? If there is such a self similar set, will it be a fractal? In order to answer the question, we have gone for generalization with continuous maps and homeomorphisms.**

Keywords: **Devaney Chaos, Fractals, Fractal Dimension, Self Similarity, Sensitivity, Topological Transitivity.**

1. INTRODUCTION

In this paper, our concern is to find out answers for the questions regarding iterated function systems, chaotic contraction maps, fractals, etc. Questions that we formulated for our investigations are;

- Do we have a chaotic iterated function system?
- Can there be chaotic contraction maps in the sense of Devaney?
- Will the chaotic contractions generate self- similar sets?
- If so will it be a fractal?

We will be answering the above given questions in the following sections. We have done two generalization here. Now we shall begin with the definition of self-similar set according to Hutchinson.

2. DEFINITIONS

Hutchinson Definition of Self-similar Set: Let K be compact set in metric space and K is self-similar set, if there exists a finite set S = {s_1,......, s_N} of contraction maps on K such that

$K = \cup_{i=1}^{N} s_i(K)$. Let f_i $(i = 1 \dots . n)$ be a family of maps defined from metric space X to itself. It is a contraction, when it hold the condition;

$d\big(f_i(x), f_i(y)\big) \leq c_i\, d(x, y),\ \ i{:}\,1 \rightarrow 0 < c < i$

Devaney Chaos: Let (X, d) be a metric space and f be a continuous mapping $f{:}\,X \rightarrow X$ which is chaotic in the sense of Devaney on X if

(1) f has sensitive dependence on initial conditions

(2) f is topologically transitive

(3) Periodic points are dense in X

Iterated Function Systems: Let U be a closed subset in an Euclidean Space. A self map f on U is a contraction, if

$$|f(x) - f(y)| \leq c|x - y|, \forall\, x, y \in U, \text{where } 0 < c < 1$$

A finite family of contractions$\{f_1, f_2, \ldots\ldots, f_n\}$, with $n \geq 2$, is called an iterated function system or IFS. We call a non-empty compact subset F of U an attractor (or invariant set) for the IFS if $F = \cup_{i=1}^{m} f_i(F)$. The unique attractor of an iterated function system is a fractal. For example, let C be the middle third Cantor set and f_1 and f_2 be a real valued self map which is defined as $f_1(x) = \frac{1}{3}x$; $f_2(x) = \frac{1}{3}x + \frac{2}{3}$. Then $f_1(C)$ and $f_2(C)$ are left and right halves of C, then $C = f_1(C) \cup f_2(C)$. Thus C is an attractor of the IFS consisting of the contractions $\{f_1, f_2\}$, which represent the basic self-similarities of the Cantor set.

Attractor: Let A admits an open neighbour U such that $\varphi^k(s) \to A$, as $\to \infty, \forall\, s \in k(X) \subset U$. Maximal open set with this property is called the basin of attraction of A, denoted by A.

Symmetric: F is symmetric if g $\in F = g^{-1} \in F$

Transitive: F is transitive if for any two non-empty open subsets $U, V \in X\ \exists n \in F^{+}$, a semi group such that $h(U) \cap V \neq \emptyset$.

Sensitive: F is sensitive on X if $\exists\, \delta > 0$ such that $\forall\, x \in X$ and $\forall r > 0, \exists\, y \in B_r(x), \omega \in (1,2,3\ldots\ldots n)$ and $n \in Z_+$ with $d\left(f_\omega^n(x), f_\omega^n(y)\right) > \delta$.

Strongly Transitive: F is strongly transitive if $O^- F = X\ \ \forall x \in X$.

Minimal: F is minimal if $\{\overline{O_F^+}(x) = X, \forall\, x \in X$

$\{O_F^+(x) = \{h(x) |\, h \in F^+\}$

$\{O_F^+(x) = \{h^{-1}(x) |\, h \in F^+\}$

Stable Set:

$$\text{(a) } W^s(x) = \{y \in X \,|\, f^n(x) \to y \text{ as } n \to +\infty\}$$

$$W^u(x) = \{y \in X \,|\, f^n(x) \to y \text{ as } n \to -\infty\}$$

(b) x is attracting fixed point of f, if $W^s(x)$ contains an open neighbourhood B of x. Then B is called the basin of attraction of x.

(c) x is repelling, $W^u(x)$ contains an open neighbourhood B of x. Then B is called the basin of repulsion of x.

$R(F) = \{h \in F^+ /\, h$ has atleast one repelling fixed point$\}$

$A(F) = \{h \in F^+ /\, h$ has atleast one attracting fixed point$\}$

Problem: Searching for iterated function system which is Devaney chaotic

Example: Consider a classical Cantor set A, where $A = [0,1/3] \cup [2/3,1]$

The IFS for the attractor is $F = (R, f_1(x) = \frac{1}{3}x, f_2(x) = \frac{1}{3}x + \frac{2}{3})$

$A = f_1(A) \cup f_2(A)$, A is a self-similarity set.

Now define $f: A \to A$ by

$$f(x) = \begin{cases} 3x, & if\ x \in f_1(A) \\ 3x-2, & if\ x \in f_2(A) \end{cases}$$

Then we can prove that $f: A \to A$ is a chaotic dynamical system in the sense of Devaney. ie. If we take any two open subsets J and K, $\exists k > 0$ such that $f^k(J) \cap K \neq \emptyset$

Now we shall find iterated function system (IFS) for continuous using tent map and logistic map. We define, IFS $F = ([0,1], 1/2 - 1/2\,x, 1/2 + 1/2\,x)$. Then $[0,1]$ is an attractor and the tent map defined from $[0,1] \to [0,1]$ is Devaney chaotic.

Consider a periodic function f defined from $s^1 \to s^1$ with period $2\pi/3$

Now define a IFS, $F = (s^1, f_1(\theta) = \theta/3, f_2(\theta) = \theta/3 + 2\pi, f_3(\theta) = \theta/3 + 4\pi/3)$. So s^1 is an attractor which is chaotic for continuous map.

3. CHAOTIC CONTRACTION

Our primary focus is to answer the question that whether a contraction is chaotic or not.

Suppose $f_i's$ are Devaney chaotic,then the questions to be answered are given below:

- Will it be a contraction?
- If they are contractions, will there exist a self-similar set?
- If there exist a self-similar set, can it be a fractal?

The answer is no, contractions cannot be chaotic.

RESULT 3.1. Contraction maps cannot be sensitive

Proof: Let $\{f_i\}_{i=1}^{n}$ be IFS for the metric space. Then $d\big(f_i(x), f_i(y)\big) \leq c_i\, d(x,y)$, where $i = 1, 2, \ldots\ldots. n, 0 < c < i$. The condition for a map to be sensitive; if for every point $x \in X$ and every positive number ε, there exists a point $y \in X$ with $d(x,y) < \varepsilon$ and positive integer n such that $d(f^n(x), f^n(y)) \geq \delta$. If we take the n^{th} iteration of the contraction mapping the constant c_i cannot be greater than δ, where $c_i's$ varies from 0 to 1. So contractive maps cannot be sensitive.

4. GENERALIZATION TO A TOPOLOGICAL SPACE

So we generalize the condition to a topological space, $T_2, f_1, f_2, \ldots\ldots f_n$ be a family of continuous maps from X to X. Define a map $\varphi: K(X) \rightarrow K(X)$, where $K(X)$ is a space of compact subsets of X with Vieterious topology, then $\varphi(A) = \cup_{i=1}^{n} (f_i(A), A \in K(X)$. Any compact subset of a compact set is open in Vieterious topology. We have to find existence of such an A defined in the above equation. If such an A exists, that must be a self-similar set defined by Hutchinson. Then X must be a complete metric space and $f_i's$ are contractions, then this A is a self-similar set.

RESULT 4. 1. If A is an attractor then A is seperable.

Proof: Since A is an attractor, all $\varphi^k(s)$ tends to A as $k \rightarrow \infty$ and contained in A. If we take the union of all these $\varphi^k(s)$, also will contain in A. ie. $\cup_0^\infty \varphi^k(s) \subset A$.

Since $\cup_0^\infty \varphi^k(s)$ is a countable dense subset of A.Then the attractor A is separable.

Example:

Now we shall see an example for non metrizable space, $X = (0,1] \times \{0\} \cup [0,1) \times \{1\}$, where X is compact, first countable and non metrizable.

Define an IFS on this space say;

$$\omega_1(x,j) = (\frac{x}{2}, j)$$

$$\omega_2(x,j) = (x + 1/2, j)$$

$\omega_3(x,j) = (1 - x, 1 - j)$, where $j = \{0,1\}$

Now,

$$\omega_1(1/2,0) = (1/4,0)$$
$$\omega_1(3/4,1) = (3/8,1)$$
$$\omega_1(3/4,0) = (3/8,0)$$
$$\omega_2(1/2,0) = (3/8,0)$$
$$\omega_2(3/4,1) = (7/8,0)$$
$$\omega_2(3/4,0) = (3/8,0)$$
$$\omega_3(1/2,0) = (1/2,1)$$
$$\omega_3(3/4,1) = (1/4,0)$$
$$\omega_3(3/4,0) = (1/4,1).$$

Now we compute the IFS for X. Then we get,

$$\omega_1((0,1],0) = ((0,1/2],0)$$

$$\omega_1([0,1),1) = ([0,1/2),1)$$

$$\omega_2((0,1],0) = ((0,3/2],0)$$

$$\omega_2([0,1),1) = ([0,3/2),1)$$

$$\omega_3((0,1],0) = ((0,1],1)$$

$$\omega_3([0,1),1) = ([0,1),0) .$$

Then $X = \omega_1(X) \cup \omega_2(X) \cup \omega_3(X)$. Then X is a self-similar set.

RESULT 4. 2. Let X be a metric space, then non-expansive map $d\big(f_i(x), f_i(y)\big) \le d(x,y)$ if $A \subset B(A)$. Then A is an attractor.

Proof: Let $d\big(f_i(x), f_i(y)\big) \le d(x,y)$and if $A \subset B(A)$. Now the Basin of A,

$B(A) = \{y \in X \rightarrow f^n(y) \rightarrow x, for\ some\ x \in A\}$. If $A \subset B(A)$, then every $x \in A, f^n(x)$ converges to some $y \in A$. $ie.\$f^n(A) \rightarrow A \Longrightarrow f(A) = A$. Therefore A is invariant and $f^n(A) \rightarrow A$. Therefore A is an attractor.

5. GENERALIZATION USING HOMEOMORPHIC MAPS

Now we shall see another generalization in compact metric space with homeomorphic maps to study the nature of an IFS.

Let X be a compact metric space. $F = \{f_1, f_2 \dots \dots f_n\}$ homeomorphism on X. Define F^+ as semigroup generated by F. The usual definition of $F^+ = \{f^i\}_{i=1}^n$which is a classical IFS. But here, the definition of F^+ is different from the classical definition.

RESULT 5.1.

If F is strongly transitive but not minimal then F is sensitive.

Proof: F is strongly transitive $\Longrightarrow \overline{O_F^-(x)} = X$, $\forall\ x \in X$, F is not minimal $\Longrightarrow \exists\ y \in X$ such that $\overline{O_F^+(x)} \ne X$. Let $z \in X \backslash \overline{O_F^+}$. Take $V = B_\delta(z)$, where $\delta = \frac{d(z, \overline{O_F^+(x)}}{4}$. Let $x \in X$ and U is a proper subset of X. Since F is strongly transitive, $\exists\ T_1, T_2 \dots \dots . . T_n \in F^+$ such that

(1) $X \subseteq \cup_{i=1}^l T_i\ (U)$

(2) $X \subseteq \cup_{i=1}^l T_j\ (T_i(U))$

RESULT 5.2.

If F is strongly transitive and $R(F) \ne \emptyset$, then F is sensitive on X.

Proof: Suppose F is strongly transitive and $h \in R(F)$. Let q be the repelling fixed point of h. Assume F is not sensitive. Then for each $n \in N$, there exists a non-empty open subset U_n of X such that $diam\left(f_n^i(U_n)\right) < \frac{1}{n}, \forall\omega \in$ {1,2,3,......n} $\forall i \in N$. Since q is attracting point, $\exists n_0 \in N$ such that $B_{n_o}(q) \subset B$. Let $n > n_o$, since F is strongly transitive $\exists T \in F^+$, such that $T^{-1}(q) \in U_n$.

$$\Longrightarrow B_{\delta_o}(q) \subset T(U_n\ )$$

Take $y \in B_\delta(q) \backslash \{q\}$, $y' = T^{-1}(y)$ and $q' = T^{-1}(q)$. Then $y', q' \in T^{-1}(y') = T^{-1}\big(B_\delta(q)\big) \in U_n$. Therefore $d(h\ o\ T(y'), h^m\ o\ T(q')) = d(h^m(y), q) \ge 1\backslash n_o > 1\backslash n$. For some $m \in\in N$, which is a contradiction. So F is sensitive.

REFERENCES

[1] G. Chen and Y. Huang, `` *Chaotic Maps: Dynamics, Fractals and Rapid Fluctuations*, 1st ed., USA: Morgan and Claypool Publishers, 2011.

[2] J. R. Munkres, *Topology*, 2nd ed., USA: Prentice Hall, 1975.

[3] L. S. Block and W. A. Coppel, *Dynamics in One Dimension,* Lecture Notes in Mathematics,no.1513,1st ed., New York: Springer – Verlag, 1992.

[4] R.L. Devaney, `` *An Introduction to Chaotic Dynamical Systems*," 2nd ed., USA: Addison-Wesley, 1989.

[5] J. E. Hutchinson, `` Fractals and Self- Similarity," $Indiana. Univ. Math. J.$, vol. 30, no. 5, pp. 713-747, Sep. 1981.